§15.4

$$\iint_R f(x,y)\,dA = \iint_G f(r,\theta)\, r\,dr\,d\theta$$

or

$$r\,d\theta\,dr$$

$R \longrightarrow G$

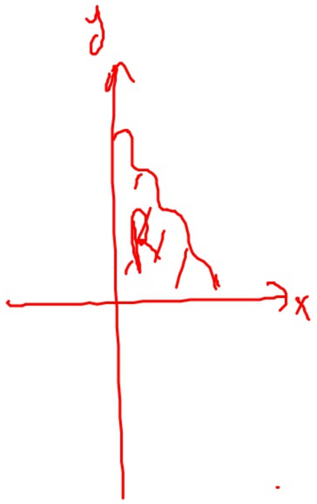
$x = r\cos\theta$

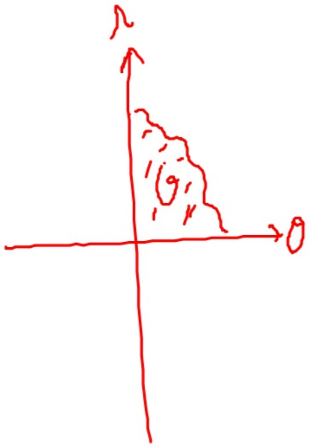
$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$= r\,d\theta\,dr$

$r \geq 0$



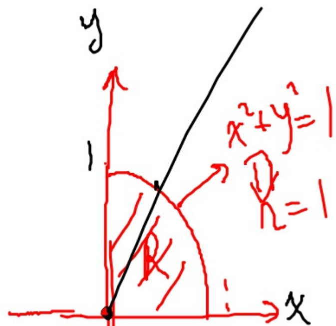


10) $$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$$

$$= \int_0^{\pi/2} \underbrace{\int_0^1 r^3\,dr}_{1/4}\,d\theta = \frac{1}{4}\,\frac{\pi}{2} = \frac{\pi}{8}$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

$$\left(\frac{r^4}{4}\right)_0^1$$

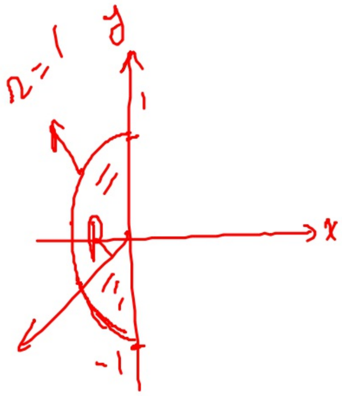


$0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

ex)

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,dx\,dy$$



$$= 4\int_{\pi/2}^{3\pi/2}\int_0^1 \frac{r^2}{1+r^2}\,dr\,d\theta$$

$$= 4\int_{\pi/2}^{3\pi/2} \boxed{\int_0^1 \left(1-\frac{1}{1+r^2}\right) dr\,d\theta}$$

$$= 4\int_{\pi/2}^{3\pi/2} \underbrace{\Big[r-\tan^{-1} r\Big]_0^1}_{1-\frac{\pi}{4}}\, d\theta = 4\left(1-\frac{\pi}{4}\right)\pi$$

$$= (4-\pi)\pi$$

$0 \le r \le 1$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

example 3

$$\iint_R e^{x^2+y^2}\,dA$$

$$= \int_0^{\pi} \boxed{\int_0^2 e^{r^2}\, r\,dr}\,d\theta$$

$$\int_0^2 e^{r^2}\, r\,dr$$

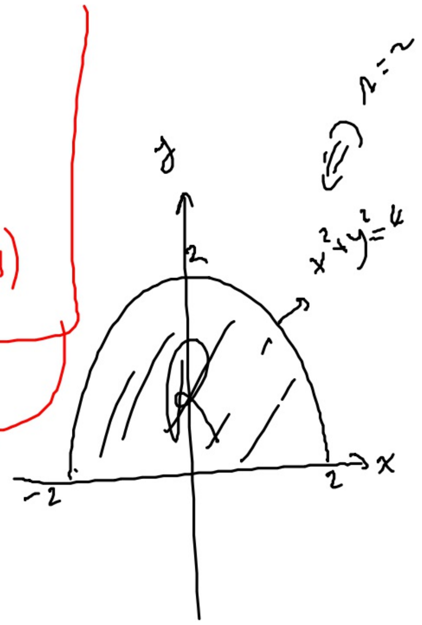
$$= \frac{1}{2}\int_0^4 e^u\,du$$

$$= \frac{1}{2}\left[e^u\right]_0^4 = \frac{1}{2}\left(e^4-1\right)$$

let $u = r^2$

$du = 2r\,dr$

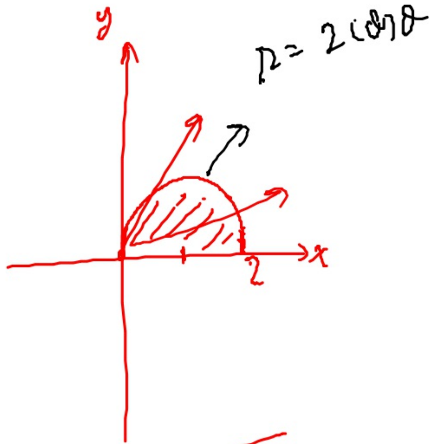
$$= \pi \int_0^2 e^{r^2}\, r\,dr = \frac{\pi}{2}\left(e^4-1\right)$$



ex

$$\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dy\,dx$$

$$\boxed{\alpha \le \theta \le \beta}$$



$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{\cancel{r^2}(\cos\theta+\sin\theta)}{\cancel{r^2}}\,dr\,d\theta$$

$$= \int_0^{\pi/2} (\cos\theta+\sin\theta)\,2\cos\theta\,d\theta \quad = \int_0^{\pi/2} (1+\cos 2\theta+\sin 2\theta)\,d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2} - \frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2} \quad = \frac{\pi}{2} + 1$$

$$y = \sqrt{1-(x-1)^2}$$

$$y^2 = 1-(x-1)^2$$

$$(x-1)^2 + y^2 = 1$$

$$x^2 + 1 - 2x + y^2 = 1$$

$$r^2 - 2r\cos\theta = 0$$

$$r = 0 \text{ or } \boxed{r = 2\cos\theta}$$

37) $\displaystyle\iint_R \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\,dA = \int_0^{2\pi}\int_1^{\sqrt{e}} \frac{\ln r^2}{\underbrace{|r|}_{=r}}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_1^{\sqrt{e}} 2\ln r\,dr$

$R:\quad 1 \le x^2+y^2 \le e$

$1 \le r^2 \le e$

$1 \le r \le \sqrt{e}$

$\displaystyle = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\,dr\,d\theta$

$\displaystyle = 2\left(r\ln r - r\right)\Big|_1^{\sqrt{e}}$

$= 2\left(\frac{\sqrt{e}}{2} - \sqrt{e} + 1\right)$

$= \left(2-\sqrt{e}\right)$

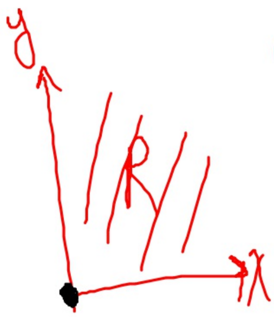
$= 2\pi\left(2-\sqrt{e}\right)$

#1) $I=\int_0^\infty e^{-x^2}\,dx$

$I=\int_0^\infty e^{-y^2}\,dy$

$$\Rightarrow I^2=\left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

$$=\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy=\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$



$$=\int_0^{\pi/2}\left[\frac{-e^{-r^2}}{2}\right]_0^\infty d\theta=\int_0^{\pi/2}\frac{1}{2}\,d\theta=\frac{\pi}{4}.$$

$$\therefore I=\frac{\sqrt{\pi}}{2}$$

7

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$$

$$= 2\int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{r}{1+r}\,dr\,d\theta$$

$$= \left(2\int_{\pi}^{3\pi/2} d\theta\right)\left(\int_{0}^{1}\left(1-\frac{1}{r+1}\right)dr\right)$$

$$= \pi\Big[r-\ln(r+1)\Big]_{0}^{1} = (1-\ln 2)\pi.$$

$$\frac{r}{r+1} = 1-\frac{1}{r+1}$$

$$\frac{r+1-1}{r+1}$$

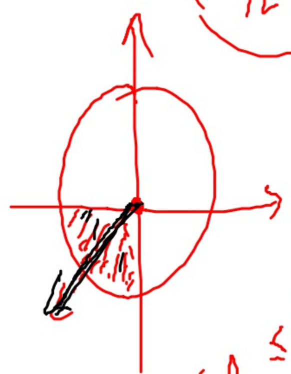
$y=-\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$

$\Downarrow$

$r=1$



$0\le r\le 1$

$\pi\le\theta\le\frac{3\pi}{2}$

22) $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx$

$y = \sqrt{2x - x^2}$

$y^2 = 2x - x^2$

$x^2 - 2x + y^2 = 0$

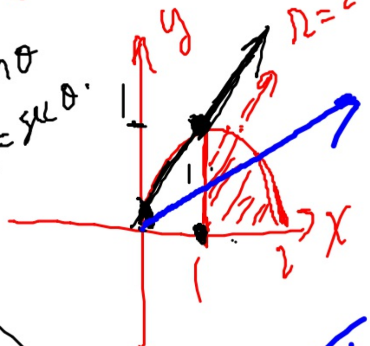
$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

$r = 2\cos\theta$

$$= \boxed{\int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-3}\, dr\, d\theta}$$

$\tan\alpha = \frac{1}{1} = 1$



$$= \int_0^{\pi/4} \left(\frac{r^{-2}}{-2} \right)_{\sec\theta}^{2\cos\theta} d\theta$$

$1 = x = r\cos\theta$

$r = \frac{1}{\cos\theta} = \sec\theta.$

$$= -\frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{4\cos^2\theta} - \frac{1}{\sec^2\theta} \right) d\theta = -\frac{1}{2} \int_0^{\pi/4} \left(\frac{\sec^2\theta}{4} - \cos^2\theta \right) d\theta = \cdots = \frac{\pi}{16}\,?$$

<u>Remark</u>: If $f(x,y) = g(x)\,h(y)$

& $\underline{R}$: $a \le x \le b$ & $c \le y \le d$

then

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$